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OSCILLATION THEOREMS FOR DIFFERENTIAL EQUATIONS OF HIGHER ORDERS AND THE SPECTRUM OF THE CORRESPONDING DIFFERENTIAL OPERATORS

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16. Abstract

Oscillatory solutions to ordinary differential equations are discussed and eight theorems are derived.

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OSCILLATION THEOREMS FOR DIFFERENTIAL EQUATIONS OF HIGHER ORDERS AND THE SPECTRUM OF THE CORRESPONDING DIFFERENTIAL OPERATORS

I. M. Glazman

There is a well-known relationship between the oscillation /423*
properties of solutions of the equation

$$l_2[y] \equiv -y'' + q(x)y = \lambda y \quad (0 \leqslant x \leqslant \infty) \qquad (1)$$

and the spectrum of any self-conjugate operator L, produced by the operation l_2. In view of this relationship, the set of points of the spectrum, which precede the point $\lambda = \lambda_0$, will be finite or infinite depending on whether Equation (1) is non-oscillatory or oscillatory when $\lambda = \lambda_0$ (i.e., whether each solution of it has a finite or infinite number of zeros).

This relationship is usually used when studying a spectrum and when certain properties of the spectrum are established, based on the non-oscillatory or oscillatory characteristics. A classic example of this is the proof of G. Weil of the spectrum discreteness characteristic [1, page 73]. Another proof of this characteristic was given by the author in [2a], using the method of disintegration introduced in this article.

In this report, the normal investigation method is transformed — namely, the spectrum is studied directly by means of disintegration — and this leads to conclusions regarding the

*Numbers in the margin indicate pagination of original foreign text.

oscillatory nature of the differential equation. It is thus not necessary to use any asymptotic properties of solutions of a differential equation.

This approach leads to the natural formulation of problems regarding the oscillation for differential equations of higher orders having the form

$$l[y] \equiv \sum_{k=0}^{n} (-1)^{n-k} [p_k(x)\, y^{(n-k)}]^{(n-k)} = \lambda y \quad (p_0(x) = 1,\ 0 \leqslant x < \infty) \tag{2}$$

or

$$(-1)^n y^{(2n)} + q(x) y = \lambda y \quad (0 \leqslant x < \infty). \tag{3}$$

Lemma 1, which is readily established by means of disintegration, lies at the basis of this discussion.

Lemma 1. Let us assume $\tilde{L}$ is a certain self-conjugate operator produced by the operation l, and let us assume U is the negative part of the spectrum for the operator $\tilde{L}$. In order that the set U be bounded below and be discrete, it is necessary and sufficient that for any $\varepsilon > 0$ there be a α, for which the square functional

$$\Phi_\varepsilon[y] \equiv \int_\alpha^\infty l[y]\, \bar{y}\, dx + \varepsilon \int_\alpha^\infty |y|^2 dx \tag{4}$$

is positive. In order that the set U be finite, it is necessary and sufficient that for a certain α the functional $\Phi_0[y]$ be positive. /424

In every case, any finite functions of $D_{\tilde{L}}$, which equal zero close to α, are assumed to be permissible for the functional $\Phi_\varepsilon[y]$.

The definition given below of an oscillatory characteristic (when n = 2, see [3]) is such that the relationship with the properties of the spectrum is retained, which was mentioned at the beginning of the article, when changing from Equation (1) to Equation (2).

Definition. Equation (2) is called oscillatory, if for any α there is a solution of this equation which has to the right of α more than one n-multiple zero. In the opposite case, Equation (2) is called non-oscillatory.

Theorem 1 may be readily established by means of the disintegration method.

Theorem 1. In order that Equation (2) be non-oscillatory when $\lambda = \lambda_0$, it is necessary and sufficient that part of the spectrum of the operator $\tilde{L}$, lying to the left of the point $\lambda = \lambda_0$, be an infinite set.

It may be shown that when Equation (2) is oscillatory the first of the n-multiple zeros of the solution given in the definition may be given arbitrarily.

The negative part of any function f(x) is designated below by $f^*(x)$, so that $f^*(x) = \min\{0, f(x)\}$.

Theorem 2. If for any $\delta > 0$ the following inequality is satisfied

$$\int\limits_{M_{k\delta}} |p_k^*(x)|\,dx < \infty \quad (k = 1, 2, \ldots, n), \tag{5}$$

where $M_{k\delta}$ is a set of the values x for which $|p_k^*(x)| \geqslant \delta$, then Equation (2) is non-oscillatory when $\lambda < 0$ (i.e., the negative part of the spectrum of the operator $\tilde{L}$ is semi-bounded below and is discrete).

In particular, the result of I. M. Rapoport [4] follows, which used asymptotic formulas for the solution of Equation (2) to establish the validity of Theorem 2, under the assumption of the summability of all coefficients $p_k(x)$ on the half-plane $x > 0$.

The fact that Equation (2) is non-oscillatory for $\lambda < 0$ also follows from Theorem 2, when the following inequality is satisfied

$$\int_0^\infty |p_k^*(x)|^{r_k} dx < \infty,$$

where $r_k \geqslant 1$ $(k = 1, 2, \ldots, n)$.

In the classical case $n = 1$, the fact that Equation (1) is non-oscillatory when $\lambda = 0$, as is known, is equivalent to the existence of a solution of the corresponding Riccati equation on a certain half-plane $[\alpha, \infty)$. In the general case, the non-oscillatory nature of Equation (2) when $\lambda = 0$ is equivalent to the existence of the solution of a certain nonlinear system of differential equations in the interval $[a, b)$ for a certain a and any $b > a$.

To formulate this system, it is sufficient to use Lemma 1 and the theorem of M. G. Kreyn [5], stipulating that in the case of a non-negative functional $\Phi_0[y]$ the operation l may be represented in the form

$$l = \mu' \mu, \tag{6}$$

where

$$\mu[y] = y^{(n)} + u_1(x)\, y^{(n-1)} + \ldots + u_n(x)\, y, \quad \mu'[y] = (-1)^n y^{(n)} + + (-1)^{n-1}[u_1(x)\, y]^{(n-1)} + \ldots + u_n(x)\, y.$$

Equating the coefficients for the derivatives $y^{(k)}$ ($k = 0, 1, \ldots, 2n - 1$) in both parts of Equation (6), we obtain the unknown system of differential equations with respect to the function $u_k(x)$ ($k = 1, 2, \ldots, n$), which may be reduced to one Riccati equation when $n = 1$ /425

$$u'^2 - u^2 - q(x) = 0. \tag{7}$$

In 1948 N. Adamov [6], studying Equation (7), established* the curvature of the set of functions $q(x)$, for which this equation has a solution on a certain half plane $[\alpha, \infty)$ (i.e., for which Equation (1) is non-oscillatory). The generalization of this fact to Equation (2) and the nonlinear system of differential equations associated with it follows directly from Lemma 1. It follows from the same lemma that, if Equation (2) is non-oscillatory, then the equation with larger coefficients is also non-oscillatory.

In the particular case of Equation (2) with constant coefficients $p_k(x) = a_k$ the set K_a of points $Q(a_1, a_2, \ldots, a_n)$ of n-dimensional space of coefficients, which correspond to Equation (2) which is non-oscillatory when $\lambda = 0$, is the closure of the set of points, for which in the sequence of 2n first main minors of the Hankel matrix $\|s_{j+k}\|_{j,k=0}^{n}$, where

*Under the assumption that $q(x)$ is periodic.

$$k\alpha_k = s_1\alpha_{k-1} - s_2\alpha_{k-2} + s_3\alpha_{k-3} - \cdots \pm s_k,\quad \alpha_{2k} = a_k,\ \alpha_{2k+1} = 0\ (k = 0, 1, \ldots, n)$$

n is alternating.

By replacing the variables

$$x = \ln t, \quad y = x^{\frac{1-2n}{2}} z \tag{8}$$

the functional $\Phi_0[y]$ corresponding to Equation (2) with constant coefficients may be reduced to the form

$$\widetilde{\Phi}_0[z] = \int_{\alpha'}^{\infty} |z_t^{(n)}|^2\, dt + \sum_{k=1}^{n} \int_{\alpha'}^{\infty} b_k t^{-2k} |z_t^{(n-k)}|^2\, dt,$$

where the numbers b_k are linear functions of the coefficients a_k

$$b_k = \varphi_k(a_1, a_2, \ldots, a_k) \quad (k = 1, 2, \ldots, n), \tag{9}$$

from which Theorem 3 follows.

Theorem 3. Let us assume the curved set K_b is the image of the set K_a determined by the transformation of (9), and let us set

$$b'_k = \liminf_{x\to\infty} p_k(x), \quad b''_k = \limsup_{x\to\infty} p_k(x) \quad (k = 1, 2, \ldots, n).$$

If $Q(b'_1, b'_2, \ldots, b'_n) \in K_b$, then Equation (2) is non-oscillatory when $\lambda = 0$. If $Q(b''_1, b''_2, \ldots, b''_n) \overline{\in} K_b$, then Equation (2) is oscillatory when $\lambda = 0$.

In particular, when n = 1 the set K_b is the half-plane $b_1 \geqslant -1/4$ (Knezer). When n = 2 the set K_b is determined by the inequalities: $b_2 \geqslant -9/4\, b_1 - 9/16$ when $b_1 \geqslant -5/2$; $b_2 \geqslant 1/4\,(2-b_1)^2$ when $b_1 \leqslant -5/2$ [3]. When n = 3 the set K_b is the portion of the space containing the first octant and bounded by the surface

$$b_1 = \varphi_1(u-2v),\quad b_2 = \varphi_2(u-2v,\ v^2-2uv),\quad b_3 = \varphi_3(u-2v,\ v^2-2uv,\ uv^2),$$

where u ≥ 0, v ≥ 0.

By iteration of the transformation (8), Theorem 3 assumes the development in the direction indicated by Hille [7] when n = 1.

In the particular case of a double-term operation, Theorem 4 holds.

Theorem 4. Equation (3) is non-oscillatory, if

$$q(x) \geqslant -\alpha_n^2 x^{-2n},$$

and oscillatory if for a certain δ > 0

$$q(x) < -(\alpha_n^2 + \delta)\, x^{-2n},$$

where the "Knezer constant" α_n^2 is determined by the formula /426

$$\alpha_n = \frac{(2n-1)!!}{2^n}.$$

Theorem 5 refines the first part of this theorem.

Theorem 5. If for any $\eta > 0$ the following inequality is satisfied

$$\int_{M_\eta} x^{2n-1} |q^*(x)| \, dx < \infty,$$

where M_η is a set of values of x for which $x^{2n} |q^*(x)| \geqslant \alpha_n^2 - \delta,$ then Equation (3) is non-oscillatory when $\lambda = 0$.

The conditions of Theorem 5, in particular, are satisfied if for a certain $r \geq 1$

$$\int_0^\infty x^{2nr-1} |q^*(x)|^r \, dx < \infty.$$

When $n = 1$, $r = 1$ the well-known non-oscillatory nature of the solution of Equation (1) follows when $\lambda = 0$ [8].

Further oscillatory conditions are obtained by using the procedure given by the author in [2b].

Theorem 6. If the function q(x) satisfies the condition

$$\int_0^\infty q(x) \, dx = -\infty,$$

then Equation (3) is oscillatory when $\lambda = 0$ (when $n = 1$ under the assumption that $q(x) \leq 0$, see [8]).

Theorem 7. If $q(x) \leq 0$ for large x and

$$\liminf_{\rho \to \infty} \rho^{2n-1} \int_\rho^\infty |q(x)| \, dx > A_n^2,$$

when

$$A_{n+1} = (2n+1)^{-1/2}\left(\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{2n-k+1}\right)^{-1} n!,$$

then Equation (3) is oscillatory when $\lambda = 0$.

Theorem 8 refines the second part of Theorem 4.

Theorem 8. If $q(x) + \alpha_n^2 x^{-2n} \leqslant 0$ for large x and

$$\liminf_{\rho \to \infty} \ln \rho \int_{\rho}^{\infty} x^{2n-1} \left| q(x) + \alpha_n^2 x^{-2n} \right| dx > B_n^2,$$

when

$$B_n^2 = \frac{n(4n^2-1)}{3 \cdot 4^{n-1}} \sum_{k=1}^{n} \frac{1}{2k-1} \sum_{k=0}^{2n-2} \frac{(-1)^k \binom{2n-2}{k}}{4n-3-k} \left[\sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n-1}{k-1}}{2n-k}\right]^{-2},$$

then Equation (3) is oscillatory when $\lambda = 0$.

We may replace $\liminf_{\rho \to \infty}$ by $\lim_{\rho_k \to \infty}$ in the latter theorems.

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